

A NOTE ON SPECIAL ORTHOGONAL GROUPS FOLLOWING WALDSPURGER

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ABSTRACT. The purpose of this note is to verify that the archimedean multiplicity one theorems shown for orthogonal groups (as well as general linear and unitary groups) in a previous paper of the authors remain valid for special orthogonal groups. The necessary ingredients to establish this variant are due to Waldspurger.

Theorem 0.1. *Let G be a special orthogonal group $\mathrm{SO}(p, q)$ or $\mathrm{SO}_n(\mathbb{C})$, $p, n \geq 1$, $q \geq 0$. Let G' be $\mathrm{SO}(p-1, q)$ or $\mathrm{SO}_{n-1}(\mathbb{C})$, viewed as a subgroup of G as usual. Then for every irreducible Casselman-Wallach smooth representation V of G , and V' of G' , one has that*

$$\dim \mathrm{Hom}_{G'}(V \widehat{\otimes} V', \mathbb{C}) \leq 1.$$

Here “ $\widehat{\otimes}$ ” stands for the completed projective tensor product of Hausdorff locally convex topological vector spaces.

We follow the general set-up of [SZ, Section 3].

Let (A, τ) be a (finite-dimensional) commutative involutive algebra over \mathbb{R} , and let E be a (non-degenerate finitely generated) Hermitian A -module, with a Hermitian form

$$\langle \cdot, \cdot \rangle_E : E \times E \rightarrow A.$$

Denote by $\mathrm{U}(E)$ the group of A -linear automorphisms of E preserving the form $\langle \cdot, \cdot \rangle_E$. Write $E_{\mathbb{R}} := E$, viewed as a real vector space. Denote by $\check{\mathrm{U}}(E)$ the subgroup of $\mathrm{GL}(E_{\mathbb{R}}) \times \{\pm 1\}$ consisting of pairs (g, δ) such that either

$$\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \quad u, v \in E,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$

This contains $\mathrm{U}(E)$ as a subgroup of index two.

First assume that (A, τ) is simple. If τ is nontrivial, we put

$$\mathrm{U}_s(E) := \mathrm{U}(E) \quad (\text{This is a general linear group or a unitary group.})$$

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and

$$\check{\check{U}}'_s(E) := \check{\check{U}}_s(E) := \check{\check{U}}(E).$$

Otherwise, τ is trivial and $A = \mathbb{R}$ or \mathbb{C} . Then we have $U(E) = O(E)$, and $\check{\check{U}}(E) = O(E) \times \{\pm 1\}$, in the usual notations. We shall put

$$U_s(E) := SO(E) \subset O(E),$$

and following Waldspurger [Wa]

$$\check{\check{U}}_s(E) := \left\{ (g, \delta) \in \check{\check{U}}(E) = O(E) \times \{\pm 1\} \mid \det(g) = \delta^{\left[\frac{\dim_A E + 1}{2}\right]} \right\}$$

and

$$\check{\check{U}}'_s(E) := \left\{ (g, \delta) \in \check{\check{U}}(E) = O(E) \times \{\pm 1\} \mid \det(g) = \delta^{\left[\frac{\dim_A E}{2}\right]} \right\}.$$

In general, write

$$(A, \tau) = (A_1, \tau_1) \times (A_2, \tau_2) \times \cdots \times (A_r, \tau_r)$$

as a product of simple commutative involutive algebras over \mathbb{R} . Then

$$(1) \quad E = E_1 \times E_2 \times \cdots \times E_r,$$

where

$$E_i := A_i \otimes_A E$$

is naturally a Hermitian A_i -module. We put

$$U_s(E) := U_s(E_1) \times U_s(E_2) \times \cdots \times U_s(E_r) \subset U(E),$$

and

$$\begin{aligned} \check{\check{U}}_s(E) &:= \check{\check{U}}_s(E_1) \times_{\{\pm 1\}} \check{\check{U}}_s(E_2) \times_{\{\pm 1\}} \cdots \times_{\{\pm 1\}} \check{\check{U}}_s(E_r) \\ &:= \{(g_1, g_2, \dots, g_r, \delta) \mid (g_i, \delta) \in \check{\check{U}}_s(E_i), i = 1, 2, \dots, r\} \\ &\subset \check{\check{U}}(E). \end{aligned}$$

The latter $(\check{\check{U}}_s(E))$ contains the former as a subgroup of index two. Denote by $\chi_{s,E}$ the quadratic character on $\check{\check{U}}_s(E)$ with kernel $U_s(E)$. Likewise we define a group $\check{\check{U}}'_s(E)$ which contains $U_s(E)$ as a subgroup of index two. Denote by $\chi'_{s,E}$ the quadratic character on $\check{\check{U}}'_s(E)$ with kernel $U_s(E)$.

Write

$$\mathfrak{u}_s(E) := \{x \in \text{End}_A(E) \mid \langle xu, v \rangle_E + \langle u, xv \rangle_E = 0, u, v \in E\}$$

for the Lie algebra of $U_s(E)$ (which is also the Lie algebra of $U(E)$). Let the groups $\check{U}_s(E)$ and $\check{U}'_s(E)$ act on $U_s(E)$ and $\mathfrak{u}_s(E)$ by

$$(2) \quad \begin{cases} (g, \delta) \cdot x := gx^\delta g^{-1}, & x \in U_s(E), \\ (g, \delta) \cdot x := \delta gxg^{-1}, & x \in \mathfrak{u}_s(E). \end{cases}$$

Also they act on E by

$$(3) \quad \begin{cases} (g, \delta) \cdot u := \delta gu, & (g, \delta) \in \check{U}_s(E), u \in E, \\ (g, \delta) \cdot u := gu, & (g, \delta) \in \check{U}'_s(E), u \in E. \end{cases}$$

It is by now standard (see for example [SZ, Section 7]) that Theorem 0.1 is implied by the first assertion of the following theorem in the case of $A = \mathbb{R}$ or \mathbb{C} , τ trivial.

Theorem 0.2. *One has that*

$$(4) \quad C_{\chi_{s,E}}^{-\infty}(U_s(E) \times E) = 0$$

and

$$(5) \quad C_{\chi_{s,E}}^{-\infty}(\mathfrak{u}_s(E) \times E) = 0.$$

Note that $(g, \delta) \mapsto (\delta g, \delta)$ is a group isomorphism from $\check{U}_s(E)$ onto $\check{U}'_s(E)$ fixing $U_s(E)$. Thus Theorem 0.2 is equivalent to

Theorem 0.3. *One has that*

$$C_{\chi'_{s,E}}^{-\infty}(U_s(E) \times E) = 0$$

and

$$C_{\chi'_{s,E}}^{-\infty}(\mathfrak{u}_s(E) \times E) = 0.$$

For E as in (1), put

$$\text{sdim}(E) := \sum_{i=1}^r \max\{\text{rank}_{A_i} E_i - 1, 0\} + \dim_{\mathbb{R}} E_{\mathbb{R}}.$$

We argue by induction on $\text{sdim}(E)$ and so will assume that Theorem 0.2 (and hence Theorem 0.3) holds whenever $\text{sdim}(E)$ is smaller.

Without loss of generality, in the remaining part of this note, assume that (A, τ) is simple and E is faithful as an A -module. Let x be a semisimple element of $U_s(E)$ or $\mathfrak{u}_s(E)$. Denote by A_x the subalgebra of $\text{End}_A(E)$ generated by A , x and x^τ . Here τ is the involution of $\text{End}_A(E)$ given by

$$\langle xu, v \rangle_E = \langle u, x^\tau v \rangle_E, \quad u, v \in E.$$

Then A_x is again a commutative involutive algebra over \mathbb{R} , and $E_x := E$ is naturally a Hermitian A_x -module. In the notations of this note, the following lemma is the first key observation of Waldspurger [Wa].

Lemma 0.4. *The group $U_s(E_x)$ is a subgroup of $U_s(E)$, the Lie algebra $\mathfrak{u}_s(E_x)$ is a Lie subalgebra of $\mathfrak{u}_s(E)$, and $\check{U}_s(E_x)$ is a subgroup of $\check{U}_s(E)$. The embeddings $U_s(E_x) \hookrightarrow U_s(E)$ and $\mathfrak{u}_s(E_x) \hookrightarrow \mathfrak{u}_s(E)$ are both $\check{U}_s(E_x)$ -equivariant. Furthermore $x \in U_s(E_x)$ if $x \in U(E)$ and $x \in \mathfrak{u}_s(E_x)$ if $x \in \mathfrak{u}(E)$.*

As in [SZ, Section 5], Harish-Chandra's method of descent and the above lemma imply the following

Proposition 0.5. *Every element of $C_{\chi_{s,E}}^{-\infty}(U_s(E) \times E)$ is supported in $(Z_E \times \mathcal{U}_E) \times E$, and every element of $C_{\chi_{s,E}}^{-\infty}(\mathfrak{u}_s(E) \times E)$ is supported in $(\mathfrak{z}_E \oplus \mathcal{N}_E) \times E$, where Z_E is the scalar multiplications (by A) in $U_s(E)$, \mathfrak{z}_E is the scalar multiplications (by A) in $\mathfrak{u}_s(E)$, \mathcal{U}_E is the set of unipotent elements of $U_s(E)$, and \mathcal{N}_E is the set of nilpotent elements of $\mathfrak{u}_s(E)$.*

By the first assertion of the above proposition, (5) will imply (4). So we only need to prove (5).

Let v be a non-degenerate element of E (i.e., $\langle v, v \rangle_E$ is invertible in A), and denote by E_v the orthogonal complement of v in E . The second key observation of Waldspurger [Wa] is the following

Lemma 0.6. *The map $(g, \delta) \mapsto (g|_{(E_v)_{\mathbb{R}}}, \delta)$ identifies the stabilizer of v in $\check{U}_s(E)$ with the group $\check{U}'_s(E_v)$. Furthermore, the restriction to $\check{U}'_s(E_v)$ of the module $\mathfrak{u}_s(E)$ is isomorphic to $\mathfrak{u}_s(E_v) \times E_v \times \mathfrak{u}_s(Av)$. Here $\mathfrak{u}_s(Av)$ carries the trivial $\check{U}'_s(E_v)$ -action.*

Again Harish-Chandra's method of descent and the above lemma imply the following

Proposition 0.7. *Every element of $C_{\chi_{s,E}}^{-\infty}(\mathfrak{u}_s(E) \times E)$ is supported in $\mathfrak{u}_s(E) \times \Gamma_E$, where $\Gamma_E := \{u \in E \mid \langle u, u \rangle_E = 0\}$ is the null cone of E .*

The (same and key) argument of [SZ, Section 4] (reduction within the null cone) works in the setting of this note and we have

Proposition 0.8. *Assume that every element of $C_{\chi_{s,E}}^{-\xi}(\mathfrak{u}_s(E) \times E)$ is supported in $(\mathfrak{z}_E \oplus \mathcal{N}_E) \times \Gamma_E$, then*

$$(6) \quad C_{\chi_{s,E}}^{-\xi}(\mathfrak{u}_s(E) \times E) = 0.$$

Here $C_{\chi_{s,E}}^{-\xi}(\mathfrak{u}_s(E) \times E)$ denotes the subspace of tempered generalized functions in $C_{\chi_{s,E}}^{-\infty}(\mathfrak{u}_s(E) \times E)$.

Now Propositions 0.5 and 0.7 imply that the hypothesis of Proposition 0.8 is satisfied. (This completes the step of reduction to the null cone.) Together with Proposition 0.8, they imply that (6) always holds. Then a general principle due to Aizenbud and Gourevitch ([AGS, Theorem 4.0.2]) implies that (5) also holds. \square

REFERENCES

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